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# Hankel continued fraction and its applications

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**Abstract.** The Hankel determinants of a given power series  $f$  can be evaluated by using the Jacobi continued fraction expansion of  $f$ . However the existence of the Jacobi continued fraction needs that all Hankel determinants of  $f$  are nonzero. We introduce *Hankel continued fraction*, whose existence and unicity are guaranteed without any condition for the power series  $f$ . The Hankel determinants can also be evaluated by using the Hankel continued fraction.

It is well known that the continued fraction expansion of a quadratic irrational number is ultimately periodic. We prove a similar result for power series. If a power series  $f$  over a finite field satisfies a quadratic functional equation, then the Hankel continued fraction is ultimately periodic. As an application, we derive the Hankel determinants of several automatic sequences, in particular, the regular paperfolding sequence. Thus we provide an automatic proof of a result obtained by Guo, Wu and Wen, which was conjectured by Coons-Vrbik.

## 1. Introduction

Let  $\mathbb{F}$  be a field and  $x$  be a parameter. We identify a sequence  $\mathbf{a} = (a_0, a_1, a_2, \dots)$  over  $\mathbb{F}$  and its generating function  $f = f(x) = a_0 + a_1x + a_2x^2 + \dots \in \mathbb{F}[[x]]$ . Usually,  $a_0 = 1$ . For each  $n \geq 1$  and  $k \geq 0$  the Hankel determinant of the series  $f$  (or of the sequence  $\mathbf{a}$ ) is defined by

$$(1.1) \quad H_n^{(k)}(f) := \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & \dots & a_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & \dots & a_{k+2n-2} \end{vmatrix} \in \mathbb{F}.$$

Let  $H_n(f) := H_n^{(0)}(f)$ , for short; the *sequence of the Hankel determinants* of  $f$  is defined to be:

$$H(f) := (H_0(f) = 1, H_1(f), H_2(f), H_3(f), \dots).$$

The Hankel determinants play an important role in the study of the irrationality exponent of automatic number. In 1998, Allouche, Peyrière,

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Wen and Wen proved that all Hankel determinants of the Thue-Morse sequence are nonzero [APWW]. Bugeaud [Bu11] was able to prove that the irrationality exponent of the Thue-Morse-Mahler number is equal to 2 by using APWW's result. Using Bugeaud's method, several authors obtained the following results: first, Coons [Co13] who proved that the irrationality exponent of the sum of the reciprocals of the Fermat numbers is 2; then, Guo, Wu and Wen who showed that the irrationality exponents of the regular paperfolding numbers are exactly 2 [GWW]. However, the evaluations of the Hankel determinants still rely on the method developed by Allouche, Peyrière, Wen and Wen, which consists of proving sixteen recurrence relations between determinants (see [APWW, Co13, GWW]). A combinatorial proof of the results by APWW and Coons about the Hankel determinants is derived by Bugeaud and the author [BH13]. In our previous paper [Ha13] short proofs of those results are presented by using Jacobi continued fraction.

The Hankel determinants of a given power series  $f$  can be evaluated by using the Jacobi continued fraction expansion of  $f$  (see, e.g., [Kr98, Kr05, Fl80, Wa48, Vi83, Ha13]). However the existence of the Jacobi continued fraction needs that all Hankel determinants of  $f$  are nonzero. In Section 2 we introduce *Hankel continued fraction*, whose existence and unicity are guaranteed without any condition for the power series. The Hankel determinants can also be evaluated by using the Hankel continued fraction (see Theorem 2.1). Let  $p$  be a prime number and  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  be the finite field of size  $p$ . In Section 3 we prove the following result.

**Theorem 1.1.** *Let  $p$  be a prime number and  $F(x) \in \mathbb{F}_p[[x]]$  be a power series satisfying the following quadratic functional equation*

$$(1.2) \quad A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

where  $A(x), B(x), C(x) \in \mathbb{F}_p[x]$  are three polynomials with one of the following conditions

- (i)  $B(0) = 1, C(0) = 0, C(x) \neq 0$ ;
- (ii)  $B(0) = 1, C(x) = 0$ ;
- (iii)  $B(0) = 1, C(0) \neq 0, A(0) = 0$ ;
- (iv)  $B(x) = 0, C(0) = 1, A(x) = -(a_k x^k)^2 + O(x^{2k+1})$  for some  $k \in \mathbb{N}$  and  $a_k \neq 0$  when  $p \neq 2$ .

Then, the Hankel continued fraction expansion of  $F(x)$  exists and is ultimately periodic. Also, the Hankel determinant sequence  $H(F)$  is ultimately periodic.

On the one hand, there is no similar result with traditional Jacobi continued fraction because of that its existence is not guaranteed, on the

other hand, it is well known that the continued fraction expansion of a quadratic irrational number is ultimately periodic. Notice that the Hankel continued fraction and the Hankel determinant sequence in Theorem 1.1 can be *entirely* calculated by Algorithm 3.3. By using Theorem 1.1 we derive the Hankel determinants of several automatic sequences.

**Theorem 1.2.** *For each pair of positive integers  $a, b$ , let*

$$(1.3) \quad G_{a,b}(x) = \frac{1}{x^{2^a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}} \in \mathbb{F}_2[[x]].$$

*Then  $H(G_{a,b})$  is ultimately periodic.*

A list of Hankel determinants for the special cases of Theorem 1.2 obtained by Algorithm 3.3 is given in Corollary 4.1. When  $a = b = 0$ , we then reprove Coons's Theorem [Co13]. The cases, where  $(a, b) = (2, 1), (2, 0), (1, 1)$ , are obtained in [Ha13] by using the Jacobi continued fraction expansion. The case, where  $a = 0$  and  $b = 2$  was conjectured by Coons and Vrbik [CV12] and recently proved by Guo, Wu and Wen [GWW] by using APWW's method. The sequence  $G_{0,2}$  is usually called *regular paperfolding sequence* [WiRP, Al87].

An ultimately periodic sequence is written in contract form by using the star sign. For instance, the sequence  $\mathbf{a} = (1, (3, 0)^*)$  represents  $(1, 3, 0, 3, 0, 3, 0, \dots)$ , that is,  $a_0 = 1$  and  $a_{2k+1} = 3, a_{2k+2} = 0$  for each positive integer  $k$ . Recall that the *Rudin-Shapiro sequence*  $(u_n)$  is defined by

$$(1.4) \quad \begin{cases} u_0 = 0, \\ u_{2n} = u_n, & u_{4n+1} = u_n, & u_{4n+3} = 1 - u_{2n+1}. \end{cases} \quad (n \geq 0)$$

**Proposition 1.3.** *Let  $(u_n)$  be the Rudin-Shapiro sequence and*

$$f_1(x) = \sum_{n \geq 0} u_{n+1} x^n; \quad f_2(x) = \sum_{n \geq 0} u_{n+2} x^n; \quad f_3(x) = \sum_{n \geq 0} u_{n+3} x^n.$$

*Then,*

$$\begin{aligned} H(f_1) &\equiv (1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1)^* \pmod{2}; \\ H(f_2) &\equiv (1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1)^* \pmod{2}; \\ H(f_3) &\equiv (1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 1, 0)^* \pmod{2}. \end{aligned}$$

Recall that Stern's sequence  $(a_n)_{n=0,1,\dots}$  is defined by (see [BV13, St58])

$$\begin{cases} a_0 = 0, & a_1 = 1 \\ a_{2n} = a_n, & a_{2n+1} = a_n + a_{n+1}. \end{cases} \quad (n \geq 1)$$

The twisted version of Stern's sequence  $(b_n)$  is defined by (see [BV13, Ba10, Al12])

$$\begin{cases} b_0 = 0, & b_1 = 1, \\ b_{2n} = -b_n, & b_{2n+1} = -(b_n + b_{n+1}). \end{cases} \quad (n \geq 1)$$

Let

$$S(x) = \sum_{n=0}^{\infty} a_{n+1}x^n \quad \text{and} \quad B(x) = \sum_{n \geq 0} b_{n+1}x^n.$$

be the generating function for Stern's sequence and twisted Stern's sequence.

**Proposition 1.4.** *The Hankel determinants of the Stern's sequence and the twisted Stern's sequence verify the following relations*

$$H_n(S)/2^{n-2} \equiv H_n(B)/2^{n-2} \equiv (0, 0, 1, 1)^* \pmod{2}.$$

The proofs of Theorem 1.2 and Propositions 1.3-4 are given in Section 4. The results obtained in the paper about Hankel determinants can be used for studying the irrationality exponent [BHWY].

## 2. Hankel continued fractions

Let  $\mathbf{u} = (u_1, u_2, \dots)$  and  $\mathbf{v} = (v_0, v_1, v_2, \dots)$  be two sequences. Recall that the *Jacobi continued fraction* attached to  $(\mathbf{u}, \mathbf{v})$ , or *J-fraction*, for short, is a continued fraction of the form

$$f(x) = \frac{v_0}{1 + u_1x - \frac{v_1x^2}{1 + u_2x - \frac{v_2x^2}{1 + u_3x - \frac{v_3x^2}{\ddots}}}}},$$

The basic properties on *J*-fractions, we now recall, can be found in [Kr98, Kr05, Fl80, Wa48, Vi83, Ha13]. The *J*-fraction of a given power series  $f$  exists if and only if all the Hankel determinants  $H_n(f)$  are nonzero. The first values of the coefficients  $u_n$  and  $v_n$  in the *J*-fraction expansion can be calculated by the *Stieltjes Algorithm*. Also, Hankel determinants can be calculated from the *J*-fraction by means of the following fundamental relation:

$$H_n(f) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}.$$

The Hankel determinants of a power series  $f$  can be calculated by the above fundamental relation if the *J*-fraction exists, which is equivalent to

the fact that all Hankel determinants of  $f$  are nonzero. In this section we define the so-called *Hankel continued fraction expansion* (*Hankel fraction* or *H-fraction*, for short) whose existence and unicity are guaranteed without any condition for the power series. The Hankel determinants can also be evaluated by using the Hankel continued fraction.

The relation between continued fractions and Hankel determinants are widely studied. See [Kr05, Vi83, Fl80] for the  $S$ - and  $J$ -fractions; [Bu10] and [Ci13] for  $C$ -fraction. The following table shows that the Hankel continued fraction has some advantage over any other type of continued fractions.

Fraction type	Parameters	Fraction existence	Fraction unicity	Hankel det. formula
$S, J$ -fraction	$\delta = 1, 2; k_j = 0$	No	Yes	Yes
$C$ -fraction	$\delta = 1, u_j(x) = 0$	Yes	Yes	No
$H$ -fraction	$\delta = 2$	Yes	Yes	Yes

*Definition 2.1.* For each positive integer  $\delta$ , a *super continued fraction* associated with  $\delta$ , called *super  $\delta$ -fraction* for short, is defined to be a continued fraction of the following form

$$(2.1) \quad F(x) = \frac{v_0 x^{k_0}}{1 + u_1(x)x - \frac{v_1 x^{k_0+k_1+\delta}}{1 + u_2(x)x - \frac{v_2 x^{k_1+k_2+\delta}}{1 + u_3(x)x - \cdots}}}$$

where  $v_j \neq 0$  are constants,  $k_j$  are nonnegative integers and  $u_j(x)$  are polynomials of degree less than or equal to  $k_{j-1} + \delta - 2$ . By convention, 0 is of degree  $-1$ .

When  $\delta = 1$  (resp.  $\delta = 2$ ) and all  $k_j = 0$ , the super  $\delta$ -fraction (2.1) is the traditional  $S$ -fraction (resp.  $J$ -fraction). A super 2-fraction is called *Hankel continued fraction*. When  $\delta = 1$  and  $u_j(x) = 0$ , the super 1-fraction is a special  $C$ -fraction (set  $b_j = k_0 + k_1 + \cdots + k_{j-1} + \lfloor j/2 \rfloor$  in [Ci13]). Notice that every power series has a unique  $C$ -fraction expansion, but not all  $C$ -fractions have Hankel determinant formula, and only those who are also super 1-fractions have.

**Theorem 2.1.** (i) Let  $\delta$  be a positive integer. Each super  $\delta$ -fraction defines a power series, and conversely, for each power series  $F(x)$ , the super  $\delta$ -fraction expansion of  $F(x)$  exists and is unique.

(ii) Let  $F(x)$  be a power series such that its  $H$ -fraction is given by (2.1) with  $\delta = 2$ . Then, all non-vanishing Hankel determinants of  $F(x)$  are given by

$$(2.2) \quad H_{s_j}(F(x)) = (-1)^\epsilon v_0^{s_j} v_1^{s_j-s_1} v_2^{s_j-s_2} \cdots v_{j-1}^{s_j-s_{j-1}},$$

where  $\epsilon = \sum_{i=0}^{j-1} k_i(k_i + 1)/2$  and  $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$  for every  $j \geq 0$ .

The first part of Theorem 2.1 is a consequence of Definition 2.1 and can be proved by using an algorithm. In fact, if  $F(x) = v_0 x^{k_0} + O(x^{k_0+1})$  with  $v_0 \neq 0$ , then,  $F(x)/(v_0 x^{k_0}) = 1 + O(x)$ . The polynomial  $u_1(x)$  can be calculated by

$$\frac{v_0 x^{k_0}}{F(x)} = 1 + u_1(x)x - x^{k_0+\delta} F_1(x).$$

We repeat the same operation for  $F_1(x)$  and get  $v_1, k_1, u_2(x)$ , etc. The second part of Theorem 2.1 follows from the next Lemma.

**Lemma 2.2.** Let  $k$  be a nonnegative integer and let  $F(x), G(x)$  be two power series satisfying

$$(2.3) \quad F(x) = \frac{x^k}{1 + u(x)x - x^{k+2}G(x)},$$

where  $u(x)$  is a polynomial of degree less than or equal to  $k$ . Then,

$$(2.4) \quad H_n(F) = (-1)^{k(k+1)/2} H_{n-k-1}(G).$$

*Proof.* Let  $F(x) = \sum_j f_j x^j$ . We have  $f_j = 0$  for  $j \leq k-1$  and  $f_k = 1$ . Let  $x^k/F(x) = \sum_j b_j x^j$  and  $G(x) = \sum_j g_j x^j$ . We have  $g_j = -b_{j+k+2}$  for  $j \geq 0$ . Let  $b_k = f_j = 0$  when  $j < 0$ . We define four matrices by

$$\begin{aligned} \mathbf{F}_1 &= (f_{i-j+k})_{0 \leq i, j \leq n-1}, \\ \mathbf{G} &= \text{Diag}((b_{i+j-k})_{0 \leq i, j \leq k}, (g_{i+j})_{0 \leq i, j \leq n-k-1}), \\ \mathbf{F} &= (f_{i+j})_{0 \leq i, j \leq n-1}, \\ \mathbf{B} &= (b_{j-i})_{0 \leq i, j \leq n-1}, \end{aligned}$$

and show that

$$(2.5) \quad \mathbf{F}_1 \times \mathbf{G} = \mathbf{F} \times \mathbf{B}.$$

For example, when  $k = 3$ ,  $n = 7$ , the four matrices and (2.5) are reproduced as follows.

$$\begin{pmatrix} 1 & . & . & . & . & . & . \\ f_4 & 1 & . & . & . & . & . \\ f_5 & f_4 & 1 & . & . & . & . \\ f_6 & f_5 & f_4 & 1 & . & . & . \\ f_7 & f_6 & f_5 & f_4 & 1 & . & . \\ f_8 & f_7 & f_6 & f_5 & f_4 & 1 & . \\ f_9 & f_8 & f_7 & f_6 & f_5 & f_4 & 1 \end{pmatrix} \begin{pmatrix} . & . & . & 1 & . & . & . \\ . & . & 1 & b_1 & . & . & . \\ . & 1 & b_1 & b_2 & . & . & . \\ 1 & b_1 & b_2 & b_3 & . & . & . \\ . & . & . & . & g_0 & g_1 & g_2 \\ . & . & . & . & g_1 & g_2 & g_3 \\ . & . & . & . & g_2 & g_3 & g_4 \end{pmatrix} \\
= \begin{pmatrix} . & . & . & 1 & f_4 & f_5 & f_6 \\ . & . & 1 & f_4 & f_5 & f_6 & f_7 \\ . & 1 & f_4 & f_5 & f_6 & f_7 & f_8 \\ 1 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \\ f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\ f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} \\ f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{15} \end{pmatrix} \begin{pmatrix} 1 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ . & 1 & b_1 & b_2 & b_3 & b_4 & b_5 \\ . & . & 1 & b_1 & b_2 & b_3 & b_4 \\ . & . & . & 1 & b_1 & b_2 & b_3 \\ . & . & . & . & 1 & b_1 & b_2 \\ . & . & . & . & . & 1 & b_1 \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

Relations (2.5) are trivial for the entry  $(i, j)$  when  $0 \leq j \leq k$  and when  $j \geq k+1, i \leq k$ . For  $i, j \geq k+1$ . The two sides of (2.5) are

$$\begin{aligned} \text{LHS} &= f_{i-1}g_{j-k-1} + f_{i-2}g_{j-k} + \cdots + f_{i-n+k-1}g_{j+n-2k-1} \\ &= -(f_{i-1}b_{j+1} + f_{i-2}b_{j+2} + \cdots + f_{i-n+k-1}b_{j+n-k+1}); \\ \text{RHS} &= f_i b_j + f_{j+1} b_{j-1} \cdots + f_{i+n-1} b_{j-n+1}. \end{aligned}$$

Since  $F(x) \sum b_j x_j = x^k$ , we have  $\text{RHS} - \text{LHS} = 0$ . Moreover,  $\det \mathbf{F} \mathbf{1} = 1$ ,  $\det \mathbf{G} = (-1)^{k(k+1)/2} H_{n-k-1}(G)$ ,  $\det \mathbf{F} = H_n(F)$ ,  $\det \mathbf{B} = 1$ . This completes the proof of (2.4).  $\square$

*Example 2.1.* Let

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1+x}}}{2x^4} \in \mathbb{Q}[[x]].$$

Then

$$f(x) = \frac{1}{1 + x - \frac{x^4}{1 - \frac{x^4}{1 + x - \frac{x^4}{1 - \frac{x^4}{1 + x - \frac{x^4}{\ddots}}}}}}.$$

In view of (2.1) we have  $v_i = 1$ ,  $k_{2i} = 0$ ,  $k_{2i+1} = 2$  for all  $i$  and  $(s_j)_{j=0,1,\dots} = (0, 1, 4, 5, 8, 9, 12, 13, \dots)$  where  $s_j$  is defined in Theorem 2.1. By Theorem 2.1 the Hankel determinant sequece is (see also [Ha13, Proposition 3.7])  $H(f) = (1, 1, 0, 0, -1, -1, 0, 0)^*$ .

*Example 2.2.* Let  $g(x)$  be the generating function for the number of distinct partitions

$$\begin{aligned} g(x) &= \prod_{n \geq 1} (1 + x^n) \in \mathbb{Q}[[x]] \\ &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \dots \end{aligned}$$

Then

$$g(x) = \frac{1}{1 - x - \frac{x^3}{1 + x + \frac{x^5}{1 - x + x^2 - x^3 - \frac{x^5}{1 + x + x^2 + \frac{x^3}{1 - x + \frac{x^3}{\ddots}}}}}}.$$

We have

$$\begin{aligned} (k_j)_{j=0,1,\dots} &= (0, 1, 2, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 2, \dots), \\ (v_j)_{j=0,1,\dots} &= (1, 1, -1, 1, -1, -1, -1, 1, -4, -1/4, 1/4, -8, \dots), \\ (s_j)_{j=0,1,\dots} &= (0, 1, 3, 6, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 21, \dots), \\ (H_j(g))_{j=0,1,\dots} &= (1, 1, 0, -1, 0, 0, -1, 0, 1, 1, 0, -1, -1, 0, 1, -4, \dots). \end{aligned}$$

*Example 2.3.* Let  $h(x) = (1 - x)^{1/3} \in \mathbb{F}_2[[x]]$ . Then

$$h(x) = \frac{1}{1 + x + \frac{x^4}{1 + x + x^2 + x^3 + \frac{x^4}{1 + x + \frac{x^8}{1 + x + x^2 + x^3 + \frac{x^{16}}{\ddots}}}}}}.$$

We have

$$\begin{aligned} (k_j)_{j=0,1,\dots} &= (0, 2, 0, 6, 8, 22, 40, \dots), \\ (v_j)_{j=0,1,\dots} &= (1, -1, -1, -1, -1, -1, -1, \dots), \\ (s_j)_{j=0,1,\dots} &= (0, 1, 4, 5, 12, 21, 44, 85, \dots), \\ (H_j(h))_{j=0,1,\dots} &= (1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, \dots). \end{aligned}$$



### 3. The periodicity

In this section we prove Theorem 1.1. Let  $\delta \in \mathbb{N}^+$  and  $\mathbb{F}$  be a field.

**Algorithm 3.1** [NextABC].

*Prototype:*  $(A^*, B^*, C^*; k, A_k, D) = \text{NextABC}(A, B, C; \delta)$

*Input:*  $A(x), B(x), C(x) \in \mathbb{F}[x]$  three polynomials such that  $B(0) = 1$ ,  $C(0) = 0, C(x) \neq 0, A(x) \neq 0$ ;

*Output:*  $A^*(x), B^*(x), C^*(x) \in \mathbb{F}[x]$ ,  $k \in \mathbb{N}^+$ ,  $A_k \neq 0 \in \mathbb{F}$ ,  $D(x) \in \mathbb{F}[x]$  a polynomial of degree less than or equal to  $k + \delta - 1$  such that  $D(0) = 1$ .

*Step 1* [Define  $k, A_k$ ]. Since  $A(x) \neq 0$ , let  $A(x) = A_k x^k + O(x^{k+1})$  with  $A_k \neq 0$ .

*Step 2.* From (1.2) we have

$$(3.1) \quad F(x) = \frac{-B + \sqrt{B^2 - 4AC}}{2C};$$

and

$$(3.2) \quad F(x) = \frac{-A(x)}{B(x) + C(x)F(x)}.$$

Using (3.1) or (3.2) to get the first terms of  $F(x)$ ,  $F(x)/(-A_k x^k)$  and of  $-A_k x^k/F(x)$ :

$$(3.3) \quad \begin{aligned} F(x) &= -A_k x^k + \dots + O(x^{2k+\delta}); \\ \frac{F(x)}{-A_k x^k} &= 1 + \dots + O(x^{k+\delta}); \\ \frac{-A_k x^k}{F(x)} &= 1 + \dots + O(x^{k+\delta}). \end{aligned}$$

*Step 3* [Define  $D$ ]. Define  $D(x), G(x)$  by

$$(3.4) \quad \frac{-A_k x^k}{F(x)} = D(x) - x^{k+\delta} G(x)$$

where  $D(x)$  is a polynomial of degree less than or equal to  $k + \delta - 1$  such that  $D(0) = 1$  and  $G(x)$  is a power series. The value of  $D(x)$  is obtained by (3.3).

*Step 4* [Define  $A^*, B^*, C^*$ ]. Let

$$(3.5) \quad \begin{aligned} A^*(x) &= (-D^2 A/A_k + B D x^k - C A_k x^{2k})/x^{2k+\delta}; \\ B^*(x) &= 2AD/(A_k x^k) - B; \\ C^*(x) &= -A x^\delta/A_k. \end{aligned}$$

We prove that  $A^*, B^*, C^*$  are polynomials in Lemma 3.2.

**Lemma 3.2.** Let  $A(x), B(x), C(x) \in \mathbb{F}[x]$  be three polynomials such that  $B(0) = 1, C(0) = 0, C(x) \neq 0, A(x) \neq 0$  and

$$(A^*, B^*, C^*; k, A_k, D) = \text{NextABC}(A, B, C; \delta)$$

obtained by Algorithm 3.1. If  $F(x)$  is the power series defined by (1.2). Then,  $F(x)$  can be written as

$$(3.6) \quad F(x) = \frac{-A_k x^k}{D(x) - x^{k+\delta} G(x)}$$

where  $G(x)$  is a power series satisfying

$$(3.7) \quad A^*(x) + B^*(x)G(x) + C^*(x)G(x)^2 = 0.$$

Furthermore,  $A^*(x), B^*(x), C^*(x)$  are three polynomials in  $\mathbb{F}[x]$  such that  $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$  and

$$(3.8) \quad \deg(A^*) \leq d; \deg(B^*) \leq d + 1; \deg(C^*) \leq d + \delta,$$

where

$$d = d(A, B, C) = \max(\deg(A) + \delta - 2, \deg(B) - 1, \deg(C) - \delta).$$

*Proof.* From (1.2) and (3.6), we have

$$A(D - x^{k+\delta}G)^2 + B(-A_k x^k)(D - x^{k+\delta}G) + C(-A_k x^k)^2 = 0.$$

Thus,  $G(x)$  satisfies

$$(3.9) \quad \bar{A}(x) + \bar{B}(x)G(x) + \bar{C}(x)G(x)^2 = 0$$

where

$$\begin{aligned} \bar{A} &= AD^2 - BA_k x^k D + CA_k^2 x^{2k}; \\ \bar{B} &= -2ADx^{k+\delta} + BA_k x^{2k+\delta}; \\ \bar{C} &= Ax^{2k+2\delta}. \end{aligned}$$

Since  $\bar{C}$  and  $\bar{B}$  are divisible by  $x^{2k+\delta}$ , so does  $\bar{A}$ . Hence, (3.5) defines three polynomials  $A^*, B^*, C^*$ . Moreover,

$$\begin{aligned} \deg(A^*) &\leq \max(\deg(A) + \delta - 2, \deg(B) - 1, \deg(C) - \delta); \\ \deg(B^*) &\leq \max(\deg(A) + \delta - 1, \deg(B)); \\ \deg(C^*) &= \deg(A) + \delta. \end{aligned}$$

Let  $d_A = \deg(A)$ ,  $d_B = \deg(B) - 1$ ,  $d_C = \deg(C) - \delta$  and  $d_A^* = \deg(A^*)$ ,  $d_B^* = \deg(B^*) - 1$ ,  $d_C^* = \deg(C^*) - \delta$ . The above inequalities become

$$\begin{aligned} d_A^* &\leq \max(d_A + \delta - 2, d_B, d_C); \\ d_B^* &\leq \max(d_A + \delta - 2, d_B); \\ d_C^* &= d_A. \end{aligned}$$

So that  $d_A^*, d_B^*, d_C^* \leq \max(d_A + \delta - 2, d_B, d_C)$ .  $\square$

**Algorithm 3.3 [HFrac].**

*Prototype:*  $(a_k, d_k, D_k)_{k=0,1,\dots} = \text{HFrac}(A, B, C; p)$

*Input:*  $p$  a prime number;

$A(x), B(x), C(x) \in \mathbb{F}_p[x]$  three polynomials such that  $B(0) = 1$ ,  $C(0) = 0$  and  $C(x) \neq 0$ ;

*Output:* a finite or infinite sequence  $(a_k, d_k, D_k)_{k=0,1,\dots}$

*Step 1.*  $j := 0$ ,  $A^{(j)} := A$ ,  $B^{(j)} := B$ ,  $C^{(j)} := C$ .

*Step 2.* If  $A^{(j)} = 0$ , then return the finite sequence  $(a_k, d_k, D_k)_{k=0,1,\dots,j-1}$ . The algorithm terminates.

*Step 3.* If  $A^{(j)} \neq 0$ , then let

$$(A^{(j+1)}, B^{(j+1)}, C^{(j+1)}; d_j, \alpha_j, D_j) := \text{NextABC}(A^{(j)}, B^{(j)}, C^{(j)}; 2).$$

Let  $j := j + 1$ .

*Step 4.* If there exists  $0 \leq i < j$  such that

$$(3.10) \quad (A^{(i)}, B^{(i)}, C^{(i)}) = (A^{(j)}, B^{(j)}, C^{(j)}),$$

then return the infinite sequence

$$(3.11) \quad ((a_k, d_k, D_k)_{k=0,1,\dots,i-1}, (a_k, d_k, D_k)_{k=i,i+1,\dots,j-1}^*).$$

The algorithm terminates. Else, go to Step 2.

*Remarks.* (i) In step 3 the conditions

$$B^{(j)}(0) = 1, C^{(j)}(0) = 0, C^{(j)}(x) \neq 0$$

are guaranteed by Lemma 3.2. Algorithm 3.1 can be applied repeatedly.

(ii) The loop Steps 2-4 will be broken at Step 2 or Step 4, since the degrees of the polynomials  $A^{(i)}, B^{(i)}, C^{(i)}$  are bounded, and the coefficients are taken from  $\mathbb{F}_p$ . The number of different triplets  $(A^{(i)}, B^{(i)}, C^{(i)})$  is finite.

*Proof of Theorem 1.1.* There are several cases to be considered. If  $B(x) \neq 0$ , then we can always suppose that  $B(x) = x^d + O(x^{d+1})$  for some  $d \in \mathbb{N}$ .

(i) If  $B(0) = 1, C(0) = 0, C(x) \neq 0$ , let

$$(a_k, d_k, D_k)_{k=0,1,\dots} = \text{HFrac}(A, B, C; p).$$

By Lemma 3.2,

$$F(x) = \frac{-a_0x^{d_0}}{D_0(x) + \frac{a_1x^{d_0+d_1+2}}{D_1(x) + \frac{a_2x^{d_1+d_2+2}}{D_2(x) + \frac{a_3x^{d_2+d_3+2}}{\ddots}}}}$$

and the above  $H$ -fraction is ultimately periodic (see Steps 2 and 4 in Algorithm 3.3). Note that if  $A(x) = 0$ , then the output sequence  $((a_k, d_k, D_k))$  ( $k = 0, 1, 2, \dots$ ) is the empty sequence. In this case  $F(x) = 0$ .

(ii) If  $B(0) = 1, C(x) = 0$ , then  $F(x) = -A(x)/B(x)$  is rational.

(iii) If  $B(0) = 1, C(0) \neq 0$  and  $A(x) = 0$ , then  $F(x)$  is rational. If  $B(0) = 1, C(0) \neq 0$  and  $A(x) = A_kx^k + O(x^{k+1})$  with  $k \geq 1$  and  $A_k \neq 0$ , then equation (1.2) has two solutions,

$$F_1(x) = \frac{-B + \sqrt{B^2 - 4AC}}{2C};$$

$$F_2(x) = \frac{-B - \sqrt{B^2 - 4AC}}{2C}.$$

Note that  $F_1(x) = -A_kx^k + O(x^{k+1})$  and  $F_2(x) = -1/C(0) + O(x)$ .

(iii.1) In the case of  $F_1(x)$ , let

$$F_1(x) = \frac{-A_kx^k}{D(x) - x^{k+2}G(x)}.$$

Then,  $G(x)$  satisfies (3.7) with polynomials  $A^*, B^*, C^*$  defined by (3.5) (see the proof of Lemma 3.2). Since  $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$ , the  $H$ -fraction expansion of  $G(x)$  exists and is ultimately periodic by case (i), so does the  $H$ -fraction expansion of  $F_1(x)$ .

(iii.2) In the case of  $F_2(x)$ , let

$$(3.12) \quad F_1(x) = \frac{-1/C(0)}{D(x) - x^2G(x)}.$$

Then,  $G(x)$  satisfies (3.7) with polynomials  $A^*, B^*, C^*$  defined (same proof as Lemma 3.2):

$$(3.13) \quad \begin{aligned} A^*(x) &= (D^2AC(0) - BD + C/C(0))/x^2; \\ B^*(x) &= -2ADC(0) + B; \\ C^*(x) &= C(0)Ax^2. \end{aligned}$$

Since  $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$ , the  $H$ -fraction expansion of  $G(x)$  exists and is ultimately periodic by case (i), so does the  $H$ -fraction expansion of  $F_2(x)$ .

(iv) If  $B(x) = 0, C(0) = 1$  (or  $C(0) \neq 0$ ) and  $A(x) = -(a_k x^k)^2 + O(x^{2k+1})$  for some  $k \in \mathbb{N}$  and  $a_k \neq 0$ , then  $F(x)$  exists

$$F(x) = \sqrt{\frac{-A(x)}{C(x)}} = \sqrt{\frac{(a_k x^k)^2 + \dots}{C(x)}} = a_k x^k \sqrt{\frac{1 + \dots}{C(x)}}$$

Let

$$F(x) = \frac{a_k x^k}{D(x) - x^{k+2} G(x)}.$$

Then,  $G(x)$  satisfies (3.7) with  $A^*, B^*, C^*$  defined (same proof as Lemma 3.2):

$$\begin{aligned} (3.14) \quad A^*(x) &= (D^2 A + C a_k^2 x^{2k})/x^{3k+2}, \\ B^*(x) &= -2ADx^{k+2}/x^{3k+2}, \\ C^*(x) &= Ax^{2k+4}/x^{3k+2}. \end{aligned}$$

If  $p \neq 2$ , then  $A^*, B^*, C^*$  are polynomials such that  $B^*(0) \neq 0, C^*(0) = 0, C^*(x) \neq 0$ . The  $H$ -fraction expansion of  $G(x)$  exists and is ultimately periodic by case (i), so does the  $H$ -fraction expansion of  $F_2(x)$ .

The periodicity of the Hankel determinant sequece  $H(F)$  is a consequence of Lamma 3.4 stated below.  $\square$

**Lemma 3.4.** *If the  $H$ -fraction expansion of a power series  $F$  is ultimately periodic, then the Hankel determinant sequece  $H(F)$  is ultimately periodic.*

*Proof.* Using the notations of Theorem 2.1 the two sequences  $(v_i)$  and  $(k_i)$  can be written as

$$\begin{aligned} (v_i) &= (v_0, v_1, \dots, v_{m-1}, (v_m, v_{m+1}, \dots, v_{m+t-1})^*); \\ (k_i) &= (k_0, k_1, \dots, k_{m-1}, (k_m, k_{m+1}, \dots, k_{m+t-1})^*). \end{aligned}$$

Let

$$\begin{aligned} \gamma_1 &= \prod_{i=m}^{m+t-1} (-1)^{k_i(k_i+1)/2}, \quad \gamma_2 = \prod_{i=m}^{m+t-1} v_i^{s_m - s_i}, \quad \gamma_3 = \prod_{i=0}^{m-1} v_i, \\ \beta &= \prod_{i=m}^{m+t-1} v_i, \quad \gamma = \gamma_1 \gamma_3^r \gamma_2 \beta^{r-s_m}, \\ r &= s_{m+t} - s_m, \quad \eta = \lceil \frac{\ell - m}{t} \rceil, \quad \rho = \ell - m - \eta t. \end{aligned}$$

For each  $\ell \geq m$  we have

$$H_{s_\ell}(F) = \prod_{i=0}^{\ell-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell-s_i}$$

and

$$(3.15) \quad H_{s_\ell+r}(F) = \prod_{i=0}^{\ell+t-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell+r-s_i}.$$

If  $p = 2$ , then  $H_{s_\ell}(F) = H_{s_\ell+r}(F) = 1$ . Hence  $H(F)$  is ultimately periodic. For general  $p$  we need evaluate (3.15).

$$\begin{aligned}
H_{s_\ell+r}(F) &= \prod_{i=0}^{\ell-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell+r-s_i} \times \prod_{i=\ell}^{\ell+t-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell+r-s_i} \\
&= H_{s_\ell}(F) \prod_{i=0}^{\ell-1} v_i^r \times \gamma_1 \prod_{i=\ell}^{\ell+t-\rho-1} v_i^{s_\ell+r-s_i} \prod_{i=\ell+t-\rho}^{\ell+t-1} v_i^{s_\ell+r-s_i} \\
&= H_{s_\ell}(F) \prod_{i=0}^{\ell-1} v_i^r \times \gamma_1 \prod_{i=\ell}^{\ell+t-\rho-1} v_i^r \prod_{i=\ell}^{\ell+t-\rho-1} v_i^{s_\ell-s_i} \prod_{i=\ell-\rho}^{\ell-1} v_i^{s_\ell-s_i} \\
&= H_{s_\ell}(F) \gamma_1 \prod_{i=0}^{\ell+t-\rho-1} v_i^r \prod_{i=\ell}^{\ell+t-\rho-1} v_i^{s_\ell-s_i} \prod_{i=\ell-\rho}^{\ell-1} v_i^{s_\ell-s_i} \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \prod_{i=m}^{\ell+t-\rho-1} v_i^r \prod_{i=\ell-\rho}^{\ell+t-\rho-1} v_i^{s_\ell-s_i} \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \prod_{i=m}^{m+\eta t+t-1} v_i^r \prod_{i=m}^{m+t-1} v_i^{s_{m+\rho}-s_i} \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \prod_{i=m}^{m+\eta t+t-1} v_i^r \prod_{i=m}^{m+t-1} v_i^{s_{m+\rho}-s_m} \times \gamma_2 \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \gamma_2^{\beta^{(\eta+1)r}} \beta^{s_{m+\rho}-s_m} \\
(3.16) \quad &= H_{s_\ell}(F) \beta^{s_\ell} \gamma.
\end{aligned}$$

We apply (3.16) recursively and get

$$H_{s_\ell+2\pi r}(F) = H_{s_\ell}(F) \beta^{2\pi s_\ell} \beta^{\pi(2\pi-1)} \gamma^{2\pi}.$$

Choose  $\pi$  such that  $\beta^\pi = 1$  and  $\gamma^{2\pi} = 1$ . Then

$$H_{s_\ell+2\pi r}(F) = H_{s_\ell}(F).$$

So that  $H(F)$  is ultimately periodic.  $\square$

The following notations are used for continued fraction

$$\mathbf{K}_{n=0}^{\infty} \frac{v_n}{u_{n+1}} = \frac{v_0}{u_1 + \frac{v_1}{u_2 + \frac{v_2}{u_3 + \frac{v_3}{u_4 + \ddots}}}}.$$

*Example (i.1).* Let  $p = 5$  and

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_5[[x]]$$

or

$$-1 + (1 - x^4)F + (-x + x^5)F^2 = 0.$$

By Algorithm 3.3 [HFrac], the power series  $F$  has the following  $H$ -fraction expansion

$$F = \frac{1}{1+4x} + \left( \frac{4x^2}{1+3x} + \frac{3x^2}{1+x} + \frac{4x^3}{1+3x+2x^2} + \frac{4x^3}{1+x} + \frac{3x^2}{1+3x} + \frac{4x^2}{1+3x} + \frac{4x^2}{1+3x} + \right)^*.$$

Hence,  $m = 1, t = 7, (k_i)_{i \geq 0} = (0, (0, 0, 1, 0, 0, 0, 0)^*)$ ,

$$(s_i)_{i \geq 0} = (0, 1, 2, 3, 5, 6, 7, 8, 9, 10, \dots),$$

$r = s_{m+t} - s_m = 8$ , and  $\beta = 4, \gamma_1 = -1, \gamma_2 = 4, \gamma_3 = 1, \gamma = 4, \pi = 2$ . So that the period is less than or equal to  $2\pi r = 32$ , starting before the index  $m = 1$ . Checking the first  $32 + 1 = 33$  terms, the period is equal to 16, starting from index 0. Hence

$$H(g) = (1, 1, 1, 2, 0, 2, 4, 1, 4, 1, 4, 2, 0, 2, 1, 1)^*.$$

*Example (i.2).* Let  $p = 2$  and

$$(3.17) \quad F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_2[[x]]$$

or

$$(3.18) \quad -1 + (1 - x^4)F + (-x + x^5)F^2 = 0.$$

By Algorithm 3.3 [HFrac], we get the following  $H$ -fraction expansion

$$F = \frac{1}{1+x} + \left( \frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \right)^*.$$

Hence,  $m = 1, t = 6, (k_i)_{i \geq 0} = (0, (0, 2, 2, 0, 0, 0)^*)$ ,

$$(s_i)_{i \geq 0} = (0, 1, 2, 5, 8, 9, 10, 11, \dots),$$

$r = s_7 - s_1 = 10$ . The period is less than or equal to 10, starting before the index  $s_m = 1$ . Checking the first  $10 + 1 = 11$  terms in  $H(f)$ , which are  $(1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, \dots)$ , we see that the period is equal to 10 and  $H(f) = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*$ .

*Example (iii.1).* Let  $p = 2$  and  $G = xF$  where  $F$  is defined in Example (i.2) by (3.17) or (3.18). We have

$$G = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2} \in \mathbb{F}_2[[x]]$$

and

$$(3.19) \quad -x + (1 - x^4)G + (-1 + x^4)G^2 = 0 \quad \text{with} \quad G(0) = 0.$$

Since the coefficient of  $G^2$  has constant term, we cannot apply Algorithm 3.3 directly. Let

$$G = \frac{x}{1 + x + x^2 + x^3 G_1}.$$

Equation (3.19) becomes

$$x^3 + (1 + x^4)G_1 + x^3 G_1^2 = 0.$$

By Algorithm 3.3 [HFrac], we get the following  $H$ -fraction expansion

$$G_1 = \frac{x^3}{1+x^4} + \left( \frac{x^6}{1} + \frac{x^4}{1+x^2} + \frac{x^4}{1} + \frac{x^6}{1+x^4} + \right)^*.$$

Hence

$$G = \frac{x}{1+x+x^2} + \left( \frac{x^6}{1+x^4} + \frac{x^6}{1} + \frac{x^4}{1+x^2} + \frac{x^4}{1} + \right)^*.$$

*Example (iii.2).* Let  $p = 2$  and

$$G = \frac{1 + \sqrt{1 - \frac{4x}{1-x^4}}}{2} \in \mathbb{F}_2[[x]]$$



and

$$(3.20) \quad -x + (1 - x^4)G + (-1 + x^4)G^2 = 0 \quad \text{with} \quad G(0) = 1.$$

Since the coefficient of  $G^2$  has constant term, we cannot apply Algorithm 3.3 directly. Let

$$G = \frac{1}{1 + x + x^2 G_1}.$$

Equation (3.20) becomes

$$(x + x^3) + (1 + x^4)G_1 + x^3 G_1^2 = 0.$$

By Algorithm 3.3 [HFrac], we get the following  $H$ -fraction expansion

$$G_1 = \frac{x}{1 + x^2} + \left( \frac{x^4}{1} + \frac{x^6}{1 + x^4} + \frac{x^6}{1} + \frac{x^4}{1 + x^2} + \right)^*.$$

Hence

$$G = \frac{1}{1 + x} + \frac{x^3}{1 + x^2} + \left( \frac{x^4}{1} + \frac{x^6}{1 + x^4} + \frac{x^6}{1} + \frac{x^4}{1 + x^2} + \right)^*.$$

*Example (iv).* Let  $p = 3$  and

$$F = \sqrt{\frac{x^2 - x^3}{1 + x^3}} \in \mathbb{F}_3[[x]]$$

or

$$(-x^2 + x^3) + (1 + x^3)F^2 = 0$$

Since the coefficient of  $F$  is zero, we cannot apply Algorithm 3.3 directly. Let

$$F = \frac{x}{1 + 2x + x^3 F_1}.$$

Equation (3.20) becomes

$$2 + (1 + x + x^2)F_1 + (2x^3 + x^4)F_1^2 = 0.$$

By Algorithm 3.3 [HFrac], we get the following  $H$ -fraction expansion

$$F_1 = \frac{1}{1 + x} + \left( \frac{x^2}{1 + x} + \frac{2x^2}{1 + x} + \frac{x^2}{1 + x} + \frac{2x^3}{1 + 2x} + \frac{2x^3}{1 + x} + \right)^*.$$

Hence, the  $H$ -fraction expansion of  $F$  is

$$\frac{x}{1 + 2x} + \frac{x^3}{1 + x} + \left( \frac{x^2}{1 + x} + \frac{2x^2}{1 + x} + \frac{x^2}{1 + x} + \frac{2x^3}{1 + 2x} + \frac{2x^3}{1 + x} + \right)^*.$$

The proof of Theorem 1.1 is also valid for super 1-fraction.

**Theorem 3.5.** *Let  $p$  be a prime number and  $F(x) \in \mathbb{F}_p[[x]]$  be a power series satisfying the following quadratic functional equation*

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

where  $A(x), B(x), C(x) \in \mathbb{F}_p[x]$  are three polynomials with one of the following conditions

(i)  $B(0) = 1, C(0) = 0, C(x) \neq 0$ ;

(ii)  $B(0) = 1, C(x) = 0$ ;

(iii)  $B(0) = 1, C(0) \neq 0, A(0) = 0$ ;

(iv)  $B(x) = 0, C(0) = 1, A(x) = -(a_k x^k)^2 + O(x^{2k+1})$  for some  $k \in \mathbb{N}$  and  $a_k \neq 0$  when  $p \neq 2$ .

Then, the super 1-fraction expansion of  $F(x)$  exists and is ultimately periodic.

#### 4. Application to automatic sequences

*Proof of Theorem 1.2.* Let  $f(x) = G_{a,b}(x) \in \mathbb{F}_2[[x]]$ . Then

$$x^{2^a} f(x) = \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};$$

$$x^{2^{a+1}} f(x^2) = \sum_{n=1}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};$$

$$x^{2^a} f(x^2) = f(x) - \frac{1}{1 - x^{2^b}};$$

$$1 + (1 + x^{2^b})f(x) + x(1 + x^{2^b})x^{2^a-1}f(x)^2 = 0.$$

The above equation is of type (1.2). By Theorem 1.1 the Hankel determinant sequence  $H(f)$  is ultimately periodic.  $\square$

The following corollary is obtained by Algorithm 3.3. The case for the regular paperfolding sequence, i.e.,  $a = 0, b = 2$ , is verified in [Section 3, Example (i.2)].

**Corollary 4.1.** *Let  $G_{a,b}(x)$  be power series in  $\mathbb{F}_2[[x]]$  defined by (1.3). Over the field  $\mathbb{F}_2$  we have*

$$H(G_{0,0}) = (1)^*;$$

$$H(G_{0,1}) = 1, 1, (0)^*;$$

$$H(G_{1,0}) = (1)^*;$$

$$H(G_{0,2}) = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*;$$

$$\begin{aligned}
H(G_{1,1}) &= (1, 1, 0, 0, 1, 1)^*; \\
H(G_{2,0}) &= (1, 1, 0, 0)^*; \\
H(G_{0,3}) &= (1^5 0^2 1^1 0^6 1^3 0^2 1^2 0^2 1^2 0^4 1^1 0^4 1^1 0^2 1^1 0^2 1^1 \\
&\quad 0^4 1^1 0^4 1^2 0^2 1^2 0^2 1^3 0^6 1^1 0^2 1^4)^*; \quad [\text{period is } 74] \\
H(G_{1,2}) &= 1, 1, 1, (0)^*; \\
H(G_{2,1}) &= (1, 1, 1, 1, 1, 1, 0, 0)^*; \\
H(G_{3,0}) &= (1, 1, 0, 0, 0, 0, 0, 0)^*; \\
H(G_{0,4}) &= (1^9 0^2 1^1 0^2 \dots 1^1 0^2 1^8)^* \quad [\text{period is } 1078]; \\
H(G_{1,3}) &= (1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1)^*; \\
H(G_{2,2}) &= (1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0)^*; \\
H(G_{3,1}) &= (1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0)^*; \\
H(G_{4,0}) &= (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^*.
\end{aligned}$$

*Proof of Proposition 1.3.* Let  $f(x) = \sum_{n \geq 0} u_n x^n \in \mathbb{F}_2[[x]]$  where  $(u_n)$  is the Rudin-Shapiro sequence defined by (1.4). Then

$$(4.1) \quad x^3 + (1+x)^4 f(x) + (1+x)^5 f^2(x) = 0.$$

Since  $u_0 = u_1 = u_2 = 0$ ,

$$f_1(x) = f(x)/x; \quad f_2(x) = f(x)/x^2; \quad f_3(x) = f(x)/x^3.$$

From (4.1) we derive

$$\begin{aligned}
x^2 + (1+x)^4 f_1(x) + (1+x)^5 x f_1^2(x) &= 0; \\
x + (1+x)^4 f_2(x) + (1+x)^5 x^2 f_2^2(x) &= 0; \\
1 + (1+x)^4 f_3(x) + (1+x)^5 x^3 f_3^2(x) &= 0.
\end{aligned}$$

Theorem 1.3 follows from Algorithm 3.3.  $\square$

*Proof of Proposition 1.4.* It is well known that [BV13]

$$(4.2) \quad S(x) = (1+x+x^2)S(x^2) \in \mathbb{Q}[[x]].$$

Since  $S(x) \pmod{2}$  is rational, there exists a positive integer  $N$  such that  $H_k(S) \equiv 0 \pmod{2}$  for all  $k \geq N$ . We must use the grafting technique, introduced in [Ha13, Section 2]. First, the  $H$ -fraction of  $S(x)$  is

$$\begin{aligned}
S(x) &= \frac{1}{1-x - \frac{x^2}{1+2x + \frac{2x^2}{1 - \frac{2x^3}{1 - 3/2x + 11/4x^2 + \dots}}}}.
\end{aligned}$$

The even number 2 occurs in the sequence  $(v_j)$ , in particular at position  $v_2$  in view of (2.1). Define  $G(x)$  by

$$S(x) = \frac{1}{1 - x - \frac{x^2}{1 + 2x + 2x^2G(x)}}.$$

From (4.2) the power series  $G(x)$  satisfies the following relation

$$(1 + x + x^2) + (1 + x + x^2)G(x) + x^4G(x^2) \equiv 0 \pmod{2}.$$

By Algorithm 3.3 we get  $H(G) \equiv (1, 1, 0, 0)^* \pmod{2}$ . By Lemma 3.2,  $H_n(S) = (-1)^n 2^{n-2} H_{n-2}(G)$ . Hence

$$H_n(S)/2^{n-2} \equiv (0, 0, 1, 1)^* \pmod{2}.$$

In the same manner,  $B(x)$  is a rational function modulo 2. We use the grafting technique. Since [BV13]

$$(4.3) \quad B(x) = 2 - (1 + x + x^2)B(x^2)$$

and

$$B(x) = \frac{1}{1 + x + \frac{x^2}{1 + \frac{2x^2}{1 - 2x + \frac{x^4}{\ddots}}}},$$

we define  $U(x)$  by

$$B(x) = \frac{1}{1 + x + \frac{x^2}{1 + 2x^2U(x)}}.$$

From (4.3) the power series  $U(x)$  satisfies the following functional equation

$$(1 + x + x^2) + (1 + x + x^2)U(x) + x^4U(x^2) \equiv 0 \pmod{2}.$$

By Algorithm 3.3 we get  $H(U) = (1, 1, 0, 0)^* \pmod{2}$ . On the other hand, the Hankel determinant  $H_n(B) = -2^{n-2}H_{n-2}(U)$  by Lemma 3.2. Hence,  $H_n(B)/2^{n-2} \equiv (0, 0, 1, 1)^* \pmod{2}$ .  $\square$

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